Investmech: Static failure criteria

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- Properties of materials
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Remember to always include the design class of the structure in failure theories. For example, multiple load paths classify a structure as **fail-safe**, and other structures are designed for **safe-life**

In **safe-life** applications, **damage tolerant** design is essential

Material properties

- Brittleness: Breaks or powders easily
- Malleability: May be pounded out into thin sheets
- Ductility: Can be plastically deformed. Material must be tough & malleable!
- Sectility: May be cut smooth with a knife
- Elasticity: Spring back to original position when stress is released
- Plasticity: Can be deformed plastically
- Flexibility: Ability to deform elastically and return to origan shape when stress sis released
- Creep
- Resilience[1]
  - Ability to absorb energy when it is deformed elastically
  - Modulus of Resilience: $U_r = \frac{\sigma_y \varepsilon_y}{2} = \frac{\sigma_y^2}{2E}$
  - References: https://en.wikipedia.org/wiki/Resilience_(materials_science) ; https://www.nuclear-power.net/nuclear-engineering/materials-science/material-properties/resilience/ ; https://www.e-education.psu.edu/matse81/node/2105
- Toughness
  - Energy that can be absorbed and still keep going
- Tenacity:
  - Reaction of materials to stress (crushing, bending, breaking, tearing). How material behaves.
  - References: https://study.com/academy/lesson/mineral-strength-tenacity-hardness-cleavage-fracture.html#:~:text=Tenacity%20refers%20to%20a%20mineral's,together%20by%20weak%20ionic%20bonds.&text=Minerals%20such%20as%20talc%20and,be%20sectile%20(like%20section).

1. https://en.wikipedia.org/wiki/Resilience_(materials_science); https://www.nuclear-power.net/nuclear-engineering/materials-

Calculation of principal stresses

- To calculate the stress in any direction use:

$$\sigma_n = l^2\sigma_x + m^2\sigma_y + n^2\sigma_z + 2(lm\tau_{xy} + nl\tau_{xz} + mn\tau_{yz})$$

- *l, m* and *n* are the direction cosines of the unit vector directing in the direction in which the stress is required
- The general form of stress transformation:

$$\sigma'_{ij} = \sum_{n=1}^{3}\sum_{m=1}^{3} l_{im} l_{jn} \sigma_{mn}$$

Where l_{im} is the direction cosine of angle between axes *i* and *m*

- $\tau_{ij} = \sigma_{ij} = \sigma_{ji}$

Stress tensor transformation

- Stress tensor transformation

$$\sigma_n = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \boldsymbol{r^T \sigma_{old} r}$$

Principal normal stresses

Where:

$$\begin{vmatrix} \sigma - \sigma_x & -\tau_{xy} & -\tau_{xz} \\ -\tau_{yx} & \sigma - \sigma_y & -\tau_{yz} \\ -\tau_{zx} & -\tau_{zy} & \sigma - \sigma_z \end{vmatrix} = 0$$

Therefore:

$$\sigma^3 - \sigma^2(\sigma_x + \sigma_y + \sigma_z) + \sigma(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{xz}^2 - \tau_{yz}^2) - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2) = 0$$

To calculate in Matlab:

E = EIG(X) is a vector containing the eigenvalues of a square matrix X. Note that X in this case is numerical stress values

Principal stress example

- Multi-axial stresses:

$$\sigma_x = 600MPa$$
$$\sigma_y = 0MPa$$
$$\sigma_z = 0MPa$$
$$\tau_{xy} = 400MPa$$
$$\tau_{xz} = 0MPa$$
$$\tau_{yz} = 0MPa$$

- Principal stresses:

» X=[600 -400 0; -400 0 0; 0 0 0];

» E=eig(X)

E =

800 = σ_1

-200 = σ_3

0 = σ_2

The principal stresses are ordered from large to small for σ_1 to σ_3

Principal axes vectors and Eigenvectors

- The Matlab command is: [V,D]=eig(X);
- D contains the eigenvalues and the columns of V represent the eigenvectors
- Example with values in previous example:

Example 1

```
>> [V,D]=eig(X)
V =
   -0.4472        0   -0.8944
   -0.8944        0    0.4472
         0   1.0000        0
D =
  -200     0     0
     0     0     0
     0     0   800
```

Example 2

```
>> X=[600 0 0; 0 200 0; 0 0 100];
>> [V,D]=eig(X)
V =
     0     0     1
     0     1     0
     1     0     0
D =
   100     0     0
     0   200     0
     0     0   600
```

Static failure theories

- Maximum Principal (Normal) Stress (Lamé)
- Maximum Shear Stress (Tresca)
- Von Mises
- Octahedral stress
- Mohr's theory
- Buckling
- Maximum principal strain (St. Venant)
- Total strain energy (Beltrami-Haigh)
- Maximum distortion energy

Maximum Principal Stress (Lamé)

- Calculate σ_1, σ_2 and σ_3

$$\sigma_N = \max(|\sigma_1|, |\sigma_2|, |\sigma_3|)$$

- Failure when:

$$\sigma_N = \sigma_u \text{ or } \sigma_o$$

- Safety factor against failure:

$$X = \frac{\sigma_u}{\sigma_N}$$

- Also used:
  - f_y - the material's yield strength
  - or f_{ut}– the material's tensile strength
- Note, some materials may have different tensile and compressive failure resistances
- SANS 10162-1 for structural steel use factored resistance: $0.9f_y$ for structural steel

Maximum Shear Stress (Tresca)

- The maximum shear stress is calculated as follows:

$$\tau_{max} \frac{\sigma_1 - \sigma_3}{2}$$

- Failure occurs when:

$$\tau_{max} \geq \frac{\sigma_f}{2}$$

$$\therefore |\sigma_1 - \sigma_2| \geq \sigma_f$$
$$|\sigma_1 - \sigma_3| \geq \sigma_f$$
$$|\sigma_2 - \sigma_3| \geq \sigma_f$$

Dowling recommends: $\sigma_f = \sigma_o$

Equation indicates that under hydrostatic stress state, the maximum shear stress is zero! In agreement with test results under compression. Under tension, use maximum principal stress criterion

σ_f is the fracture stress. The yield strength, ultimate tensile strength, or any other allowable strength may be used.

Von Mises Theory

- Equation:

$$\sigma_{vm} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$
$$= \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)}$$

  - Failure occur when:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 \geq 2\sigma_f^2$$

- In most cases the failure stress is taken as the material yield strength

Octahedral Shear Stress Theory = von Mises theory = distortion energy theory

- Equation:

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2}$$
$$= \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

- The octahedral shear stress at failure:

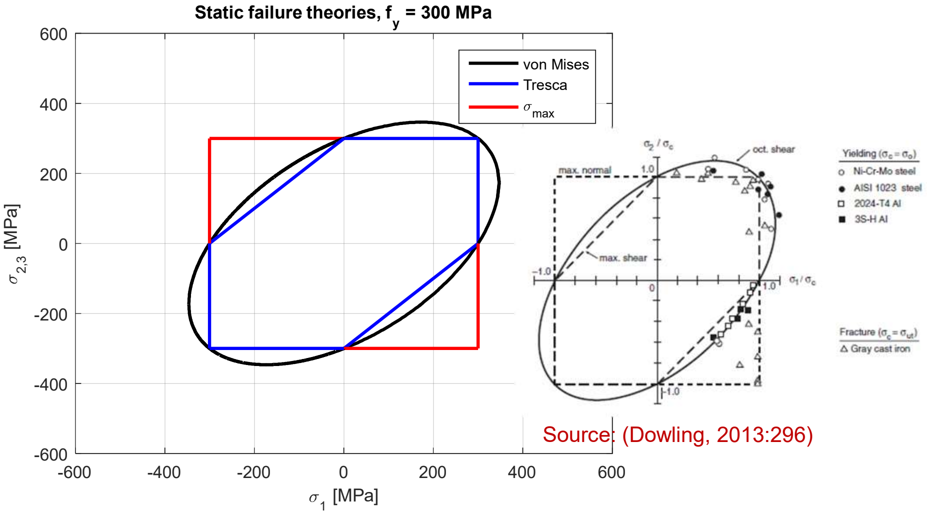
$$\tau_{oct} = S_{xy} = \frac{\sqrt{2}}{3} f_y$$

- In most cases the failure stress is taken as the material yield strength

Distortion-energy theory

- Is also called the:
  - Von Mises or von Mises-Hencky theory
  - Shear-energy theory
  - Octahedral-shear-stress theory

Static failure theories for steel with $f_y = 300\ MPa$

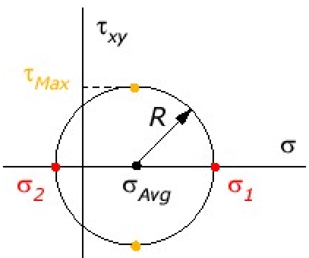


Source: (Dowling, 2013:296)

Mohr's failure theory

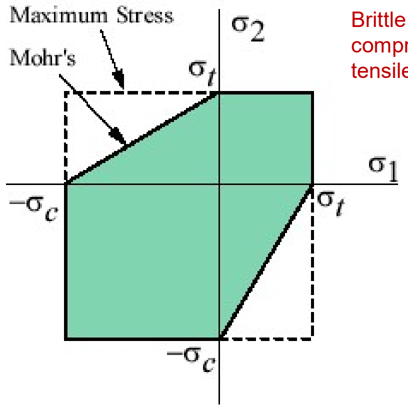
- Also known as
  - Coulomb-Mohr criterion
  - Internal friction theory
- Based on the Mohr's circle

Case	Principal stresses		Criterion requirements
1	Both in tension	$\sigma_1>0$, $\sigma_2>0$	$\sigma_1 < \sigma_t$ $\sigma_2 < \sigma_t$
2	Both in compression	$\sigma_1<0$, $\sigma_2<0$	$\sigma_1 > -\sigma_c$ $\sigma_2 > -\sigma_c$
3	σ_1 in tension, σ_2 in compression	$\sigma_1>0$, $\sigma_2<0$	$\frac{\sigma_1}{\sigma_t} + \frac{\sigma_2}{-\sigma_c} < 1$
4	σ_1 in compression, σ_2 in tension	$\sigma_1<0$, $\sigma_2>0$	$\frac{\sigma_1}{-\sigma_c} + \frac{\sigma_2}{\sigma_t} < 1$

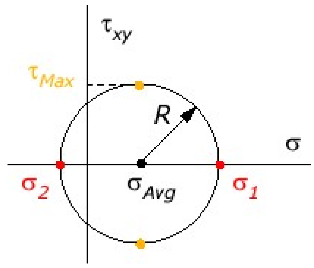


Angle of radius R with horizontal is 2θ

Mohr's vs. Maximum Stress criteria



Brittle materials have higher compressive strength than tensile strength

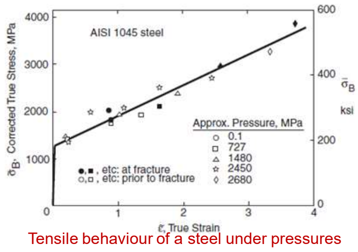


Other failure theories

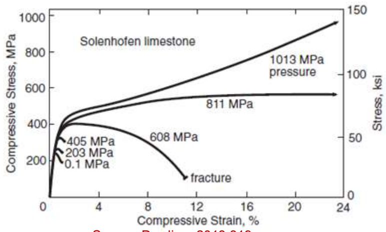
- Maximum normal strain theory (St. Venant's theory)
  - Use for:
    - ❑ Crash simulations
    - ❑ Furnace design due to refractory reheating
    - ❑ For welded detail on structural steel, Investmech uses maximum strain of 21%
- Total strain energy theory (Beltrami theory)

Brittle vs ductile behaviour

- Ductile
  - Static strength generally limited by yielding
    - Metals & Polymers
  - Mode of failure is dislocation motion
- Brittle
  - Generally limited by fracture
    - Gray cast iron & other cast metals
    - Stone, ceramics and glasses
    - Concrete
  - Do not exhibit well-defined yielding behaviour
  - Fail after small elongation ≤ 5%
    - May exhibit considerable elongation under hydrostatic compressive stress



Tensile behaviour of a steel under pressures



Source: Dowling, 2013:319

Stress-strain of limestone cylinders under axial compression subject to hydrostatic pressures

Which one do I use?

Results showed:

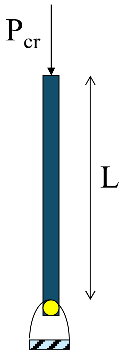
- Ductile materials (> 5% elongation @ break)
  Main yield mechanism is dislocation motion in ductile materials
  - Maximum shear stress
  - Von Mises
  - Maximum principal stress
- Brittle materials (<= 5% elongation @ break)
  Brittle material has $\sigma_{tension} \ll \sigma_{compression}$
  - Maximum principal stress (Lamé)
  - Mohr-Coulomb theory

Investmech:

- Applies maximum principal stress & von Mises theories for static design
- For fatigue, principal stresses are calculated – that is the basis for the S-N curves
- Crashworthiness, furnace design, etc (plastic deformation occurs)
  - Maximum engineering strain (e.g. 21% for welded structural steel)
  - True principal strain

Buckling

- Pinned-End column



$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Many textbooks has equations for different end supports on the critical load on columns.

Non-stress-based criteria

- Success of parts not necessarily determined by strength
  - Stiffness, vibrational characteristics, fatigue resistance, creep resistance
- Example
  - Rigidity in automotive vehicles
  - Weight reduction in bicycles
  - Patio deck – stiffness to prevent excessive deformation

Conclusion

- Always look at all failure criteria
  - The one that prevails first will be the mode in which failure will occur
- For furnaces: maximum strain of 20% for heat-up cycles
- Fatigue analysis concerns the calculation of “damage” to the structure and is the life until a detectable crack initiates
- Fracture Mechanics determines failure during the crack growth phase